SOME PARAMETRIZATION THEOREMS FOR MEASURABLE SETS WITH UNCOUNTABLE SECTIONS

BY

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ABSTRACT.

In this note we prove the following result. This improves the results of Wesley (W1) and Cenzer and Mauldin (CM).

Theorem. *Let X be an analytic space, Y a Polish space and $A \subset X \times Y$ an analytic set with sections A_x uncountable. Then there is a $B \subset A$ with sections B_x a homeomorph of the Cantor set and $B \in \mathcal{S}_X \otimes \mathcal{B}_Y$, where $\mathcal{S}_X$ denotes the smallest σ-algebra on X containing all Borel sets and closed under Souslin operations and $\mathcal{B}_Y$ the Borel σ-algebra of Y.*[1]

1. Introduction.

Using forcing, in (W1), Wesley proved the following result:

Theorem 1.1. *Let $B \subset [0,1] \times [0,1]$ a Borel set with uncountable sections B_x, $x \in [0,1]$. Then there is a bijection $f : [0,1] \times [0,1] \to B$ with $\pi_1(f(x,y)) = x$ for all (x,y), where $\pi_1 : [0,1] \times [0,1] \to [0,1]$ is the projection to the first coordinates, and such that both f and f^{-1} are Lebesgue measurable.*

Wesley was motivated by some similar problems in mathematical economics (W2). Further, such results have applications in measurable transformations on compact groups (Ch). In (CM), Cenzer and Mauldin gave the following generalization of this result. The proof of Cenzer and Mauldin is forcing free. In what follows for any analytic space X, let $\mathcal{S}_X$ denote the *Selivanowski σ-algebra*, i.e., the smallest σ-algebra on X containing all Borel sets and closed under Souslin operations.

Theorem 1.2. *Let X and Y be Polish spaces and $B \subset X \times Y$ a Borel set with all sections B_x uncountable. Then there is a bijection*

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$f : X \times Y \to B$ with $\pi_X(f(x,y)) = x$ for all (x,y) and both f and f^{-1} are measurable with respect to $\mathcal{S}_{X\times Y}$.

In this note we give some generalizations of these results. More than the generalizations, it is our proof which is of some interest. For instance, our method proves the following interesting result:

Theorem 1.3. *Let X be an analytic space, Y a Polish space and $A \subset X \times Y$ analytic with sections A_x uncountable. Then there is a $B \subset A$ with sections B_x a homeomorph of the Cantor set and $B \in \mathcal{S}_X \otimes \mathcal{B}_Y$.*

Wesley (W1) gets a weaker result using forcing, whereas Cenzer and Mauldin get a result stronger than Wesley's but weaker than ours using some recursion theory (CM, p. 406). It is known that a Borel set $B \subset X \times Y$ with sections B_x uncountable need not contain a Borel set with perfect (i.e., a homeomorph of 2^ω) sections (CM, Example, p.405). In this light, Theorem 1.3 is of some significance. The main new idea in our proof is to use the representation theorems for measurable multifunctions of the author (S2). In section 2, we give some preliminary results and in section 3, we prove our main results.

2. Preliminaries.

Let ω denote the set of all natural numbers $\{0, 1, 2, \ldots\}$ with the discrete topology. The space ω^ω of all infinite sequences of natural numbers will be equipped with the product topology. For any metric space X, $\mathcal{B}_X$ will denote the Borel σ-algebra of X and $\mathcal{C}_X$ the smallest σ-algebra on X that contains all Borel sets and all sets of first category in X. It is pertinent to recall here that on any analytic space X, the σ-algebra of universally measurable subsets of X and $\mathcal{C}_X$ are closed under Souslin operations. For other notations and terminology in Descriptive set theory we follow (S1). If X is a topological space and $A \subset X$, then $cl(A)$ will denote the closure of A. Let $(T, \mathcal{A})$ be a measurable space and Y a standard Borel space. An *$\mathcal{A}$-measurable transition probability* on $T \times Y$ is a map $P : T \times \mathcal{B}_Y \to [0,1]$ such that

(1) For every $x \in T$, $P(x, \cdot)$ is a probability measure on Y.
(2) For every Borel set B in Y, $P(\cdot, B)$ is $\mathcal{A}$-measurable.

Let $(T, \mathcal{A})$ be a measurable space and X a Polish space, i.e. a completely metrizable, second countable metric space. A closed valued multifunction $F : T \to X$ is a map with domain T and values nonempty closed sets in X. We say that F is *$\mathcal{A}$-measurable* if for every open set U in X,

$$F^{-1}(U) = \{t \in T : F(t) \cap U \neq \emptyset\} \in \mathcal{A}.$$

An ingredient in our proof is the following generalization of Von-Neumann selection theorem (S1, Theorem 5.5.8). This, of course, was implicit in the original proof of Von Neumann.

Proposition 2.1. *Let $(T, \mathcal{A})$ be a measurable space with $\mathcal{A}$ closed under Souslin operations, X a Polish space and $B \in \mathcal{A} \otimes \mathcal{B}_X$. Then $\pi_T(B) \in \mathcal{A}$ and there is an $\mathcal{A}$-measurable function $f : \pi_T(B) \to X$ whose graph is contained in B, where $\pi_T : T \times X \to T$ denotes the projection map onto T.*

The following corollary will be used without explicit mention.

Corollary 2.2. *Let T, $\mathcal{A}$, X and B be as in the Proposition 2.1. Then for every Borel set $C \subset X$, the sets*

$$\{t \in T : \exists x \in C[(t, x) \in B]\}$$

and

$$\{t \in T : \forall x \in C[(t, x) \in B]\}$$

are in $\mathcal{A}$.

Proposition 2.3. *Let T, X, B and $\mathcal{A}$ be as in corollary 2.2. Also assume that for every $t \in T$, A_t is open in X. Then we can write*

$$B = \cup_n (B_n \times U_n),$$

where $B_n \in \mathcal{A}$ and U_n open in X.

Proof. Fix a countable base $\{U_n\}$ for X and define

$$B_n = \{t \in T : \forall x \in U_n((t, x)) \in A\}.$$

By Corolloary 2.2, each $B_n \in \mathcal{A}$ and our proof is complete. $\dashv$

We shall also use the following well-known measurable analogue of Cantor-Schröder-Bernstein theorem (S1, Proposition 3.3.6).

Proposition 2.4. *Let $(T, \mathcal{A})$ and $(S, \mathcal{B})$ be measurable spaces. Suppose there exist one-to-one, bimeasurable functions $f : T \to S$ and $g : S \to T$. Then there is a bijection $h : T \to S$ which is bimeasurable.*

Theorem 2.5. (S2) *If $(T, \mathcal{A})$ is a measurable space, X a Polish space and $F : T \to X$ a closed valued, $\mathcal{A}$-measurable multifunction, then there is a map $f : T \times \omega^\omega \to X$ such that*

(1) *for every* $t \in T$*, the map* $\alpha \to f(t, \alpha)$ *from* ω^ω *to* X *is continuous and onto* $F(t)$*, and*
(2) *for every* $\alpha \in \omega^\omega$*, the map* $t \to f(t, \alpha)$ *is* $\mathcal{A}$*-measurable.*

Remark. (1) and (2) implies that f is $\mathcal{A} \otimes \mathcal{B}_{\omega^\omega}$-measurable.

Proposition 2.6. *Let* $(T, \mathcal{A})$*,* X*,* $F : T \to X$ *and* f *be as in Theorem 2.5. Also assume that each* $F(t)$ *is uncountable and* $\mathcal{A}$ *is closed under Souslin operations. Then there is a sequence* $\{g_i\}$ *of* $\mathcal{A}$*-measurable functions from* T *to* ω^ω *such that for every* $t \in T$*,* $\{g_i(t)\}$ *is dense-in-itself and for every* $i \neq j$*,* $f(t, g_i(t)) \neq f(t, g_j(t))$*.*

Proof. Fix a complete metric d on ω^ω. Let E be the set of all $(t, (\alpha_i)) \in T \times (\omega^\omega)^\omega$ satisfying the following conditions:

$$\forall i \neq j (f(t, \alpha_i) \neq f(t, \alpha_j)) \ \& \ \forall i, k \exists j \neq i (d(\alpha_i, \alpha_j) < \frac{1}{k+1}).$$

Since f is $\mathcal{A} \otimes \mathcal{B}_{\omega^\omega}$-measurable, $E \in \mathcal{A} \otimes \mathcal{B}_{(\omega^\omega)^\omega}$. Since each $F(t)$ is uncountable, $E_t \neq \emptyset$ for all t. Our result follows from Proposition 2.1. ⊣

Proposition 2.7. *Let* X *be an analytic space,* Y *a Polish space and* A *an analytic subset of* $X \times Y$ *with projection* $\pi_X(A) = X$*. Then there is a map* $h : X \times \omega^\omega \to Y$ *such that*
(1) *for every* $x \in X$*,* $\alpha \to h(x, \alpha)$ *is a continuous map from* ω^ω *onto* A_x*, and*
(2) *for every* $\alpha \in \omega^\omega$*,* $x \to h(x, \alpha)$ *is* $\mathcal{S}_X$*-measurable.*

Proof. Since A is analytic, there is a continuous map f from ω^ω onto A. Define

$$G = \{(x, \alpha) \in X \times \omega^\omega : \pi_X(f(\alpha)) = x\}.$$

Clearly G is closed and the closed valued multifunction $x \to G_x$ is $\mathcal{S}_X$-measurable. Therefore, by Theorem 2.5, there is a map $g : X \times \omega^\omega \to \omega^\omega$ such that for every $x \in X$, $\alpha \to g(x, \alpha)$ is continuous and onto G_x and for every $\alpha \in \omega^\omega$, $x \to g(x, \alpha)$ is $\mathcal{S}_X$-measurable. We now take $h = f \circ g$. ⊣

Corollary 2.8. *Let* X*,* Y*,* A *and* h *be as in Proposition 2.7. Further assume that each* A_x *is uncountable. Then there is a sequence* $\{g_i\}$ *of* $\mathcal{S}_X$*-measurable functions from* X *to* ω^ω *such that for every* $x \in X$*,* $\{g_i(x)\}$ *is dense-in-itself and for every* $i \neq j$*,* $h(x, g_i(x)) \neq h(x, g_j(x))$*.*

To prove this we argue as in the proof of Proposition 2.6.

3. Parametrization Theorems.

The following result plays an important role in this article.

Theorem 3.1. *Let X be an analytic space, Y a Polish space and $A \subset X \times Y$ an analytic set such that for every $x \in X$, the section A_x is uncountable. Then there is a map $u : X \times 2^\omega \to Y$ such that*

(1) *for every $x \in X$, the map $\alpha \to u(x, \alpha)$ defined on 2^ω is one-to-one, continuous and into A_x, and*
(2) *for every $\alpha \in 2^\omega$, the map $x \to u(x, \alpha)$ is $\mathcal{S}_X$-measurable.*

We shall give a proof of this result at the end of this section. Before that we draw some corollaries from this result.

Theorem 3.2. *Let X, Y and $A \subset X \times Y$ be as in Theorem 3.1. Then there is a $\mathcal{S}_X$-measurable transition probabilty $P : X \times \mathcal{B}_Y \to [0, 1]$ such that for every $x \in X$, $P(x, \cdot)$ is a continuous probability on Y with support contained in A_x.*

Proof. Let u be as in Theorem 3.1 and λ denote the Lebesgue measure on 2^ω. Define a transition probability $Q : X \times 2^\omega \to [0, 1]$ by

$$Q(x, B) = \lambda(B), \; x \in X, B \in \mathcal{B}_{2^\omega}.$$

Define

$$P(x, B) = Q(x, u(x, \cdot)^{-1}(B)), \; x \in X, B \in \mathcal{B}_Y.$$

Let $B \subset Y$ be Borel and $C = u^{-1}(B)$. By the remark made after Theorem 2.5, $C \in \mathcal{S}_X \otimes \mathcal{B}_Y$. We need to prove that $x \to \lambda(C_x)$ is $\mathcal{S}_X$-measurable. So, our result will be proved if we show that for every $D \in \mathcal{S}_X \otimes \mathcal{B}_{2^\omega}$, $x \to \lambda(D_x)$ is $\mathcal{S}_X$-measurable. This is certainly true for every set $D \in \mathcal{S}_X \otimes \mathcal{B}_{2^\omega}$ of the form $E \times F$. Further, the set of all $D \in \mathcal{S}_X \otimes \mathcal{B}_{2^\omega}$ for which our claim is true form a σ-algebra. This completes the proof. $\dashv$

Theorem 3.3. *Let X an analytic space, Y a Polish space and B a Borel set in $X \times Y$ with sections $B_x = \{y \in Y : (x, y) \in B\}$ uncountable for every $x \in X$. Then there is a bijection $f : X \times Y \to B$ with $\pi_X(f(x, y)) = x$ for all (x, y) and both f and f^{-1} $\mathcal{S}_X \otimes \mathcal{B}_Y$-measurable.*

Proof. By Borel isomorphism theorem (RS), without any loss of generality, we assume that $Y = 2^\omega$. Let $u : X \times 2^\omega$ be obtained as in Theorem 2.5 with $A = B$. Then u is $\mathcal{S}_X \otimes \mathcal{B}_{2^\omega}$-measurable. Define $f : X \times 2^\omega \to X \times 2^\omega$ by

$$f(x, \delta) = (x, u(x, \delta)), \; x \in X, \delta \in 2^\omega.$$

Then f is $\mathcal{S}_X \otimes \mathcal{B}_{2^\omega}$-bimeasurable. We only need to show that for every $D \in \mathcal{S}_X \otimes \mathcal{B}_{2^\omega}$, $f(D) \in \mathcal{S}_X \otimes \mathcal{B}_{2^\omega}$. To see this, first note that since f is one-to-one, it suffices to show that for every $B \in \mathcal{S}_X$ and every open U in 2^ω, $f(B \times U) \in \mathcal{S}_X \otimes \mathcal{B}_{2^\omega}$. Set $C = u^{-1}(U)$. Then $C \in \mathcal{S}_X \otimes \mathcal{B}_{2^\omega}$ with C_x open for all $x \in X$. By Proposition 2.3, there is a sequence $\{C_n\}$ of sets in $\mathcal{S}_X$ and a sequence $\{V_n\}$ of clopen sets in 2^ω such that $C = \cup_n (C_n \times V_n)$. Thus, we have proved our claim. It follows that the set

$$B = \{(x, y) \in X \times 2^\omega : \exists \alpha \in 2^\omega (y = f(x, \alpha))\} \in \mathcal{S}_X \otimes \mathcal{B}_{2^\omega}.$$

Let $g : B \to X \times 2^\omega$ denote the inclusion map. Clearly g is $\mathcal{S}_X \otimes \mathcal{B}_{2^\omega}$-bimeasurable. The result now follows from Proposition 2.4. $\dashv$

Remark. As is shown in the above proof, the range of f is in $\mathcal{S}_X \otimes \mathcal{B}_Y$. So, Theorem 1.3 follows.

By carrying out the standard construction of a Cantor set inside an uncountable analytic set we get the proof of Theorem 3.1.

Proof of Theorem 3.1.

Let $\mathcal{A} = \mathcal{S}_X$. Take h as in Proposition 2.7 and g_i's as in Corollary 2.8. We now fix a countable base $\{U(m)\}$ containing the whole space and a complete metric $d < 1$ for ω^ω. We also fix a complete metric ρ on Y. We claim that for each finite sequence $(n_0, \cdots, n_{k-1})$ of 0's and 1's including the empty sequence e, there exist $\mathcal{A}$-measurable functions $i(n_0, \cdots, n_{k-1}) : X \to \omega$ and $m(n_0, \cdots, n_{k-1}) : X \to \omega$ such that for each k, each $(n_0, \cdots, n_{k-1})$ and each $x \in X$ the following conditions are satisfied:

(1) $g_{i(n_0,\cdots,n_{k-1})(x)}(x) \in U(m(n_0, \cdots, n_{k-1})(x))$.
(2) d-diameter$U(m(n_0, \cdots, n_{k-1})(x)) < 2^{-k}$
and ρ-diameter$(h(x, U(m(n_0, \cdots, n_{k-1})(x)))) < 2^{-k}$.
(3) $cl(U(m(n_0, \cdots, n_{k-1}, n)(x))) \subset U(m(n_0, \cdots, n_{k-1})(x))$ for $n = 0, 1$.
(4) For each $(n'_0, \cdots, n'_{k-1}) \neq (n_0, \cdots, n_{k-1})$,

$$cl(h(x, U(m(n'_0, \cdots, n'_{k-1})(x)))) \cap cl(h(x, U(m(n_0, \cdots, n_{k-1})(x)))) = \emptyset.$$

To construct these sets, we proceed by induction on k. Let $i(e)(x) = 0$ and $m(e)(x) = m$, where $U(m) = \omega^\omega$. Let $i(0)(x) = 1$ and $i(1)(x) = 2$ for every $x \in X$. For each pair (m_0, m_1) of natural numbers with d-diameters of $U(m_0)$, $U(m_1)$ less than $1/2$ and their closures disjoint, let $E(m_0, m_1)$ be the set of all $x \in X$ satisfing the conditions

$$g_1(x) \in U(m_0), g_2(x) \in U(m_1),$$

$$\rho - \text{diameters of } h(x, U(m_0)), h(x, U(m_1)) < 1/2$$

and

$$cl(h(x, U(m_0))) \cap cl(h(x, U(m_1)))) = \emptyset.$$

Then $E(m_0, m_1) \in \mathcal{A}$ and $\cup_{m_0, m_1} E(m_0, m_1) = X$. Take pairwise disjoint sets $D(m_0, m_1) \subset E(m_0, m_1)$ in $\mathcal{A}$ whose union is X. We define $m(0)(x) = m_0$ and $m(1)(x) = m_1$ if $x \in D(m_0, m_1)$.

We now show the next stage of the construction after which our claim will be easily seen. Fix a pair of natural numbers (m_0, m_1) such that $D(m_0, m_1) \neq \emptyset$. Fix $\delta = 0, 1$. So, for $x \in D(m_0, m_1)$, $m(\delta)(x) = m_\delta$. For each $i_0 \neq i_1$, define

$$F(i_0, i_1) = \{x \in D(m_0, m_1) : g_{i_0}(x), g_{i_1}(x) \in U_{m_\delta}\}.$$

Then $F(i_0, i_1) \in \mathcal{A}$ and their union is $D(m_0, m_1)$. Let $F'(i_0, i_1) \subset F(i_0, i_1)$ be pairwise sets in $\mathcal{A}$ whose union is $D(m_0, m_1)$. We define $i(\delta, 0)(x) = i_0$ and $i(\delta, 1)(x) = i_1$ if $x \in F'(i_0, i_1)$.

Fix a pair (i_0, i_1) with $F'(i_0, i_1) \neq \emptyset$. Note that for $x \in F'(i_0, i_1)$, $i(\delta, 0)(x) = i_0$ and $i(\delta, 1)(x) = i_1$. For each pair (m_0, m_1) of natural numbers with d-daimeters of $U(m_0)$ and $u(m_1)$ less than 1/4 and their closures disjoint, let $G(m_0, m_1)$ be the set of all $x \in F'(i_0, i_1)$ satisfying the following conditions:

$$g_{i_0}(x) \in U(m_0), g_{i_1}(x) \in U(m_1),$$

$$cl(U(m_0)), cl(U(m_1)) \subset U(m(\delta)(x)),$$

$$\text{diameters of } h(x, U(m_0)), h(x, U(m_1)) < 1/4$$

and

$$cl(h(x, U(m_0))) \cap cl(h(x, U(m_1)))) = \emptyset.$$

Then $G(m_0, m_1) \in \mathcal{A}$ and their union is $F'(i_0, i_1)$. Let $G'(m_0, m_1)$ be pairwise disjoint sets in $\mathcal{A}$ with union $F'(i_0, i_1)$. We define $m(\delta, 0)(x) = m_0$ and $m(\delta, 1)(x) = m_1$ for $x \in G'(m_0, m_1)$. Proceeding similarly, we shall complete our construction.

Take any $x \in X$ and $\alpha \in 2^\omega$. Note that $\{h(x, g_{i(\alpha|k)(x)}(x))\}$ is a Cauchy sequence in Y, where $\alpha|k$ is the restriction of α to k, whose limit we define to be $u(x, \alpha)$. Our result is easily seen now. $\dashv$

Here are a few generalizations of our results.

Theorem 3.4. *Let $(T, \mathcal{A})$ be a measurable space with $\mathcal{A}$ closed under Souslin operations and Y a Polish space. Suppose $B \in \mathcal{A} \otimes \mathcal{B}_Y$ with sections B_t uncountable. Then there is a bijection $f : T \times Y \to B$ such that $\pi_T(f(t, y)) = t$ for all (t, y) and both f and f^{-1} are $\mathcal{A} \otimes \mathcal{B}_Y$-measurable.*

Proof. Since $B \in \mathcal{A} \otimes \mathcal{B}_Y$, there is a countably generated sub σ-algebra $\mathcal{A}'$ of $\mathcal{A}$ such that $B \in \mathcal{A}' \otimes \mathcal{B}_Y$. Fix a countable generator $\{A_n\}$ for $\mathcal{A}'$ and define $\chi : T \to 2^\omega$ by

$$\chi(t) = (\chi_{A_n}(t)),\ t \in T.$$

Then $\chi : (T, \mathcal{A}') \to (\chi(T), \mathcal{B}_{\chi(T)})$ is a Borel isomorphism and $\chi'(B)$ is Borel in $\chi(T) \times Y$, where $\chi'(t, y) = (t, \chi(t, y))$, $(t, y) \in T \times Y$. Let C be a Borel set in $2^\omega \times Y$ such that $B = C \cap (\chi(T) \times Y)$ and

$$T' = \{\alpha \in 2^\omega : C_\alpha \text{ is uncountable}\}.$$

Then T' is analytic (S1, Proposition 4.3.7). By Theorem 3.3, there is a bijection $f' : T' \times Y \to C \cap (T' \times Y)$ such that $\pi_{T'}(f'(t, y)) = t$ and both f' and $(f')^{-1}$ are $\mathcal{S}_{T'} \otimes \mathcal{B}_Y$ measurable. Now define $f(t, y) = f'(\chi'(t, y))$, $(t, y) \in T \times Y$. Since $\mathcal{A}$ is closed under Souslin operations, f has the desired properties. $\dashv$

Using the same idea, we can generalize Theorem 3.1 as follows:

Theorem 3.5. *Let $(T, \mathcal{A})$ be a measurable space and Y a Polish space. Let $A \subset T \times Y$ be a set whose sections A_t are uncountable and which is obtained by performing Souslin operation on a system of sets in $\mathcal{A} \otimes \mathcal{B}_Y$. Let $\mathcal{S}$ denote the smallest σ-algebra that contains $\mathcal{A}$ and which is closed under Souslin operations. Then there is a map $f : T \times 2^\omega \to Y$ such that for each $t \in T$, $f(t, \cdot)$ is one-to-one, continuous and into A_t and for each $\alpha \in 2^\omega$, $f(\cdot, \alpha)$ is $\mathcal{S}$-measurable. There is also a bijection $g : T \times Y \to A$ such that $\pi_T(g(t, y)) = t$ and both g and g^{-1} are measurable with respect to $\mathcal{A} \otimes \mathcal{D}$, where $\mathcal{D}$ is the σ-algebra generated by all analytic sets in Y.*

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